Jacobi Elliptic Function Solutions of Three Coupled Nonlinear Physical Equations

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The Jacobi elliptic function solutions of coupled nonlinear partial differential equations, including the coupled modified KdV (mKdV) equations, long-short-wave interaction system and the Davey-Stewartson (DS) equations, are obtained by using the mixed dn-sn method. The solutions obtained in this paper include the single and the combined Jacobi elliptic function solutions. In the limiting case, the solitary wave solutions of the systems are also given. — PACS: 02.30.Jr; 03.40.Kf; 03.65.Fd

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1. Introduction

Nonlinear evolution equations (NLEEs) are special classes of nonlinear partial differential equations (NLPDEs) which have been studied intensively in recent decades. Searching for exact solutions of NLPDEs in mathematical physics attracts considerable interest. Several important direct methods have been developed for obtaining travelling wave solutions to NLEEs such as the inverse scattering method [1], the tanh-function method [2], the extended tanh-function method [3] and the homogeneous balance method [4]. A symbolic software package to find exact solutions of NLPDEs has been described [5 – 6]. Also, various methods were presented to seek the periodic wave solutions, expressed by Jacobi elliptic functions (JEFs), of some NLEEs such as the JEF expansion method [7], the improved Jacobian elliptic function method [8], the sinh-Gordon equation expansion method [9], the extended Jacobian elliptic function expansion method [10], the mapping method [11, 12], the F-expansion method [13] and other methods [14]. Fan and Zhang [15] have extended the JEF method and have obtained doubly periodic wave solutions of special-type NLEEs. Recently, the sinh-Gordon equation expansion method has been extended to seek JEF solutions of the $(2+1)$-dimensional long-wave-short-wave resonance interaction equation [16]. Various exact solutions were obtained by these methods, including the solitary wave solutions, shock wave solutions and periodic wave solutions.

Very recently, we have proposed the mixed dn-sn method [17] to obtain various exact solutions in terms of JEFs to some nonlinear wave equations. The basic idea of the method is as follows: For a given NLPDE, say in two independent variables,

$$F(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0. \tag{1.1}$$

Let $u(x,t) = u(\xi)$; $\xi = x - \omega t$, where ω is the wave speed, (1.1) may be reduced to an ordinary differential equation (ODE)

$$G(u, u', u'', ...) = 0, \quad u' = \frac{\mathrm{d}u}{\mathrm{d}\xi}. \tag{1.2}$$

We search for the solution of (1.2) in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{N} A_i W^i + \sqrt{a^2 - W^2} \sum_{i=0}^{N-1} b_i W^i, \tag{1.3}$$

where N is a positive integer, which can be determined by balancing the highest order linear term with the nonlinear term(s) in (1.2), and a, A_i, b_i are constants to be determined. If the balancing number N is not a positive integer, we can introduce a transformation $u = v^N$ and turn equation (1.1) into another equation for v, whose balancing number will be a positive integer. If G is not a polynomial of u and its various derivatives, then we may introduce an appropriate transformation to turn the equation into the differential polynomial type.

We introduce $W = W(\xi)$ which satisfies the elliptic equation

$$W' = \sqrt{(a^2 - W^2)(W^2 - a^2(1-m))}. \quad (1.4)$$

The solutions of (1.4) are given by

$$\begin{aligned} W &= a\,\mathrm{dn}(a(x-\omega t)|m), \\ W &= a\sqrt{1-m}\,\mathrm{nd}(a(x-\omega t)|m), \end{aligned} \quad (1.5)$$

where $\mathrm{dn}(a\xi|m)$ and $\mathrm{nd}(a\xi|m) = 1/\mathrm{dn}(a\xi|m)$ are the JEFs with modulus m $(0 < m < 1)$.

Substituting (1.3) into (1.2) yields the algebraic equation

$$P(W) + \sqrt{a^2 - W^2}\,Q(W) = 0,$$

where $P(W)$ and $Q(W)$ are polynomials in W. Setting the coefficients of the various powers of W in P and Q to zero will yield a system of algebraic equations in the unknowns A_i, b_i, a, ω and m. Solving this system, we can determine these unknowns. Therefore, we can obtain several classes of exact solutions involving the JEFs sn, dn and nd, cd functions, where $\mathrm{cd}(a\xi|m) = \mathrm{cn}(a\xi|m)/\mathrm{dn}(a\xi|m)$, cn is the Jacobi cnoidal function. If $b_i = 0$, $i = 0, 1, 2, \ldots, N-1$, then (1.3) constitutes the dn (or nd) expansions.

The JEFs $\mathrm{sn}(a\xi|m)$, $\mathrm{cn}(a\xi|m)$ and $\mathrm{dn}(a\xi|m)$ are double periodic and have the following properties:

$$\begin{aligned} &\mathrm{sn}^2(a\xi|m) + \mathrm{cn}^2(a\xi|m) = 1, \\ &\mathrm{dn}^2(a\xi|m) + m\,\mathrm{sn}^2(a\xi|m) = 1. \end{aligned}$$

In the limit $m \longrightarrow 1$, the JEFs degenerate to the hyperbolic functions, i. e.,

$$\begin{aligned} &\mathrm{sn}(a\xi|1) \longrightarrow \tanh(a\xi), \\ &\mathrm{cn}(a\xi|1) \longrightarrow \mathrm{sech}(a\xi), \\ &\mathrm{dn}(a\xi|1) \longrightarrow \mathrm{sech}(a\xi). \end{aligned}$$

Detailed explanations about JEFs can be found in [18].

In this paper, the mixed dn-sn method will be used to construct the exact solutions, in terms of JEFs, for three coupled nonlinear evolution equations in $(1+1)$-dimensional and $(2+1)$-dimensional space. The rest of the paper is organized as follows: In Section 2, we obtain abundant JEF solutions to the coupled modified KdV (mKdV) equations, the long-short-wave interaction system and the Davey-Stewartson (DS) equations. To show the properties of the obtained solutions, we draw plots for some elliptic function solutions (see Figs. 1, 2). Finally, we conclude the paper in Section 3.

2. Applications to Coupled Systems

2.1. The Coupled mKdV Equations

Consider the coupled mKdV equations [8]

$$\begin{aligned} &u_t + \alpha v_x + \beta u^2 u_x + \delta u_{xxx} = 0, \\ &v_t + r(uv)_x + s v v_x = 0, \end{aligned} \quad (2.1)$$

where α, β, δ, r, and s are constants. Here the mixed dn-sn method will give some new exact solutions to the coupled mKdV equations (2.1). In order to find travelling wave solutions of (2.1), we let

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \quad \xi = x - \omega t. \quad (2.2)$$

Then, (2.1) becomes

$$\begin{aligned} &-\omega U' + \alpha V' + \beta U^2 U' + \delta U''' = 0, \\ &-\omega V' + r(UV)' + s V V' = 0. \end{aligned} \quad (2.3)$$

Balancing U''' with U^2U' and VV' with $(UV)'$ leads to the following ansatz, respectively:

$$\begin{aligned} U(\xi) &= A_0 + A_1 W + b_0\sqrt{a^2 - W^2}, \\ V(\xi) &= B_0 + B_1 W + b_1\sqrt{a^2 - W^2}. \end{aligned} \quad (2.4)$$

Substituting (2.4) into (2.3) yields the following system of algebraic equations:

$$\begin{aligned} &[\beta A_1^2 - 3\beta b_0^2 - 6\delta)]A_1 = 0, \quad A_0[A_1^2 - b_0^2] = 0, \\ &[-\omega + \beta A_0^2 + \beta b_0^2 a^2 + \delta(2-m)a^2]A_1 + \alpha B_1 = 0, \\ &[3\beta A_1^2 - \beta b_0^2 - 6\delta)]b_0 = 0, \quad b_0 A_0 A_1 = 0, \\ &[\omega - \beta A_0^2 - \beta b_0^2 a^2 + 2\beta A_1^2 a^2 - \delta(5-m)a^2]b_0 - \alpha b_1 = 0, \\ &2r(A_1 B_1 - b_0 b_1) + s(B_1^2 - b_1^2) = 0, \\ &-\omega B_1 + r(A_0 B_1 + A_1 B_0) + s B_0 B_1 = 0, \\ &r(b_1 A_1 + b_0 B_1) + s b_1 B_1 = 0, \\ &\omega b_1 - r(b_1 A_0 + b_0 B_0) - s b_1 B_0 = 0. \end{aligned}$$

Solving these equations, we get the following solutions that include three different cases:

Case 1: $b_0 = 0$,

$$\begin{aligned} &b_1 = A_0 = 0, \quad A_1 = \pm\sqrt{6\delta/\beta}, \\ &B_0 = \frac{2\delta a^2 s(2-m) - 4\alpha r}{s^2}, \\ &B_1 = \mp\frac{2r}{s}\sqrt{6\delta/\beta}, \\ &\omega = -\frac{2\alpha r - \delta a^2 s(2-m)}{s}. \end{aligned} \tag{2.5}$$

Case 2: $A_1 = 0$,

$$\begin{aligned} &B_1 = A_0 = 0, \quad b_0 = \pm\sqrt{-6\delta/\beta}, \\ &B_0 = -\frac{2\delta a^2 s(1+m) + 4\alpha r}{s^2}, \\ &b_1 = \mp\frac{2r}{s}\sqrt{-6\delta/\beta}, \\ &\omega = -\frac{2\alpha r + \delta a^2 s(1+m)}{s}. \end{aligned} \tag{2.6}$$

Case 3: $A_0 = 0$,

$$\begin{aligned} &A_1 = \pm\sqrt{3\delta/2\beta}, \quad b_0 = \pm\sqrt{-3\delta/2\beta}, \quad B_0 = -\frac{\delta a^2 s(2m-1) + 4\alpha r}{s^2}, \\ &B_1 = \mp\frac{r}{s}\sqrt{6\delta/\beta}, \quad b_1 = \mp\frac{r}{s}\sqrt{-6\delta/\beta}, \quad \omega = -\frac{4\alpha r + \delta a^2 s(2m-1)}{2s}, \end{aligned} \tag{2.7}$$

with a being an arbitrary constant.

Substituting (2.5)–(2.7) into (2.4) and using the special solutions (1.5) of equation (1.4), we obtain the following JEF solutions of (2.1):

$$\begin{aligned} &u_1(x,t) = \pm a\sqrt{6\delta/\beta}\,\mathrm{dn}\left(a\left(x + \frac{2\alpha r - \delta a^2 s(2-m)}{s}t\right)\Big|m\right), \\ &v_1(x,t) = \frac{2\delta a^2 s(2-m) - 4\alpha r}{s^2} \mp \frac{2ra}{s}\sqrt{6\delta/\beta}\,\mathrm{dn}\left(a\left(x + \frac{2\alpha r - \delta a^2 s(2-m)}{s}t\right)\Big|m\right), \quad \beta\delta > 0, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &u_2(x,t) = \pm a\sqrt{6\delta(1-m)/\beta}\,\mathrm{nd}\left(a\left(x + \frac{2\alpha r - \delta a^2 s(2-m)}{s}t\right)\Big|m\right), \\ &v_2(x,t) = \frac{2\delta a^2 s(2-m) - 4\alpha r}{s^2} \mp \frac{2ra}{s}\sqrt{6\delta(1-m)/\beta}\,\mathrm{nd}\left(a\left(x + \frac{2\alpha r - \delta a^2 s(2-m)}{s}t\right)|m\right), \quad \beta\delta > 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &u_3(x,t) = \pm a\sqrt{-6m\delta/\beta}\,\mathrm{sn}\left(a\left(x + \frac{2\alpha r + \delta a^2 s(1+m)}{s}t\right)\Big|m\right), \\ &v_3(x,t) = -\frac{2\delta a^2 s(1+m) + 4\alpha r}{s^2} \mp \frac{2ra}{s}\sqrt{-6m\delta/\beta}\,\mathrm{sn}\left(a\left(x + \frac{2\alpha r + \delta a^2 s(1+m)}{s}t\right)\Big|m\right), \quad \beta\delta < 0, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &u_4(x,t) = \pm a\sqrt{-6m\delta/\beta}\,\mathrm{cd}\left(a\left(x + \frac{2\alpha r + \delta a^2 s(1+m)}{s}t\right)\Big|m\right), \\ &v_4(x,t) = -\frac{2\delta a^2 s(1+m) + 4\alpha r}{s^2} \mp \frac{2ra}{s}\sqrt{-6m\delta/\beta}\,\mathrm{cd}\left(a\left(x + \frac{2\alpha r + \delta a^2 s(1+m)}{s}t\right)|m\right), \quad \beta\delta < 0, \end{aligned} \tag{2.11}$$

and the combined JEF solutions

$$\begin{aligned} &u_5(x,t) = \pm a\sqrt{3\delta/(2\beta)}\left[\mathrm{dn}(a\xi\,|m) + \mathrm{i}\sqrt{m}\,\mathrm{sn}(a\xi\,|m)\right], \\ &v_5(x,t) = -\frac{\delta a^2 s(2m-1) + 4\alpha r}{s^2} \mp \frac{ra}{s}\sqrt{6\delta/\beta}\left[\mathrm{dn}(a\xi\,|m) + \mathrm{i}\sqrt{m}\,\mathrm{sn}(a\xi\,|m)\right], \end{aligned} \tag{2.12}$$

$$\begin{aligned} &u_6(x,t) = \pm a\sqrt{3\delta/(2\beta)}\left[\sqrt{1-m}\,\mathrm{nd}(a\xi\,|m) + \mathrm{i}\sqrt{m}\,\mathrm{cd}(a\xi\,|m)\right], \\ &v_6(x,t) = -\frac{\delta a^2 s(2m-1) + 4\alpha r}{s^2} \mp \frac{ra}{s}\sqrt{6\delta/\beta}\left[\sqrt{1-m}\,\mathrm{nd}(a\xi\,|m) + \mathrm{i}\sqrt{m}\,\mathrm{cd}(a\xi\,|m)\right]. \end{aligned} \tag{2.13}$$

with $\xi = x + \frac{4\alpha r + \delta a^2 s(2m-1)}{2s}t$ and $\beta\delta > 0$,

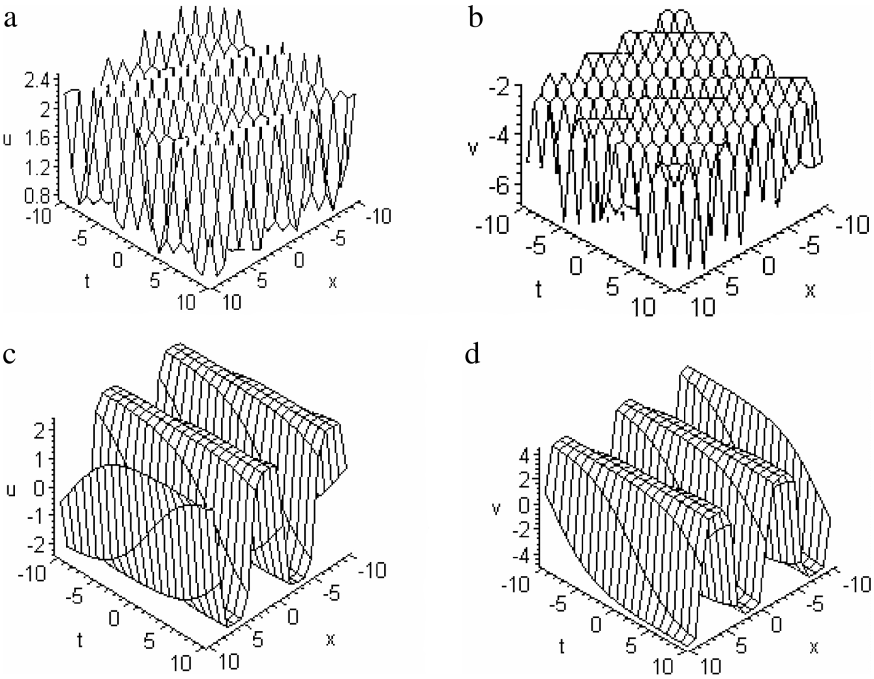

Fig. 1. The solutions u_1 (a) and v_1 (b) in (2.8) with parameters $\alpha=\beta=\delta=r=s=a=1$, $m=0.95$ and the solutions u_3 (c) and v_3 (d) in (2.10) with $\alpha=\beta=r=s=a=1$, $\delta=-1$, $m=0.9$.

The solutions (2.8) and (2.10) are the same as the results obtained in [8] by means of the improved Jacobian elliptic function method. With $m\longrightarrow 1$ in (2.8) and (2.10), the solitary wave solutions to (2.1) given in [8] are also recovered. Compared with the results in [8], here we further find some new solutions (2.9), (2.11)–(2.13). As $m\longrightarrow 1$, the solutions (2.12) degenerate to

$$\begin{aligned} u_5(x,t) &= \pm a\sqrt{\frac{3\delta}{2\beta}}\left[\operatorname{sech}(a\xi)+\mathrm{i}\tanh(a\xi)\right],\\ v_5(x,t) &= -\frac{\delta a^2 s+4\alpha r}{s^2}\\ &\quad\mp\frac{ra}{s}\sqrt{\frac{6\delta}{\beta}}\left[\operatorname{sech}(a\xi)+\mathrm{i}\tanh(a\xi)\right], \end{aligned} \tag{2.14}$$

To show the properties of the JEF solutions, we draw plots for the solutions u_1, v_1 and u_3, v_3 (see Fig. 1).

2.2. The Long-short-wave Interaction System

Consider the long-short-wave interaction system [19, 20]

$$\mathrm{i}\psi_t+\psi_{xx}-\psi v=0,\quad v_t+v_x+(|\psi|^2)_x=0, \tag{2.15}$$

where $\psi(x,t)$ is a complex function and $v(x,t)$ is a real function. Wang *et al.* [20] have used the F-expansion method and obtained periodic wave solutions for the system (2.15). In this paper, we try to deal with the system (2.15) by the mixed dn-sn method and give some new exact solutions. Let

$$\begin{aligned} &\psi(x,t)=\mathrm{e}^{i\eta}u(x,t),\ \eta=\alpha x+\beta t,\\ &v(x,t)=V(\xi),\ u(x,t)=U(\xi),\ \xi=x-\omega t, \end{aligned} \tag{2.16}$$

where α and β are constants and $u(x,t)$ is a real function. Substituting (2.16) into (2.15), we find $\omega=2\alpha$ and U, V satisfy the following coupled ODEs:

$$\begin{aligned} &U''-UV-(\alpha^2+\beta)U=0,\\ &(1-2\alpha)V'+2UU'=0. \end{aligned} \tag{2.17}$$

The mixed dn-sn method gives

$$\begin{aligned} &U(\xi)=A_0+A_1W+b_0\sqrt{a^2-W^2},\\ &V(\xi)=B_0+B_1W+(b_1+b_2W)\sqrt{a^2-W^2}. \end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.17) yields

$$\begin{aligned} &(2+B_2)A_1-b_0b_2=0,\ A_0B_2+A_1B_1-b_0b_1=0,\\ &[a^2(2-m)-B_0-(\alpha^2+\beta)]A_1-A_0B_1-b_0b_2a^2=0,\\ &(B_0+\alpha^2+\beta)A_0+b_0b_1a^2=0,\\ &b_0(2+B_2)+A_1b_2=0,\quad A_0b_2+A_1b_1+b_0B_1=0, \end{aligned}$$

$$b_0[a^2(1-m)-B_0-(\alpha^2+\beta)]-A_0b_1=0,\quad (1-2\alpha)B_2+A_1^2-b_0^2=0,$$
$$(1-2\alpha)B_1+2A_0A_1=0,\quad (1-2\alpha)b_2+2b_0A_1=0,\quad (1-2\alpha)b_1+2A_0b_0=0,$$

from which, with aid of Maple, we find three kinds of solutions, namely,

$$A_0=b_1=b_2=B_1=b_0=0,\quad A_1=\pm\sqrt{2(1-2\alpha)},\quad B_0=a^2(2-m)-(\alpha^2+\beta),\quad B_2=-2, \tag{2.19}$$

$$A_0=A_1=b_1=b_2=B_1=0,\quad b_0=\pm\sqrt{2(2\alpha-1)},\quad B_0=a^2(1-m)-(\alpha^2+\beta),\quad B_2=-2, \tag{2.20}$$

with a,α and β being arbitrary constants and

$$A_0=b_1=B_1=0,\, A_1=\pm\sqrt{\frac{1-2\alpha}{2}},\, b_0=\pm\sqrt{\frac{2\alpha-1}{2}},\, B_0=a^2(1-m)-(\alpha^2+\beta),\, B_2=-1,\, b_2=\mp\mathrm{i}. \tag{2.21}$$

Substituting (2.19)–(2.21) into (2.18) and using the special solutions of (1.4), we obtain the following exact solutions expressed by JEFs of (2.15):

$$\begin{aligned}\psi_1(x,t)&=\pm a\sqrt{2(1-2\alpha)}\,\mathrm{e}^{\mathrm{i}\eta}\,\mathrm{dn}(a(x-2\alpha t)|m),\quad \alpha<\frac{1}{2},\\ v_1(x,t)&=a^2(2-m)-(\alpha^2+\beta)-2a^2\mathrm{dn}^2(a(x-2\alpha t)|m),\end{aligned} \tag{2.22}$$

$$\begin{aligned}\psi_2(x,t)&=\pm a\sqrt{2(1-m)(1-2\alpha)}\,\mathrm{e}^{\mathrm{i}\eta}\,\mathrm{nd}(a(x-2\alpha t)|m),\quad \alpha<\frac{1}{2},\\ v_2(x,t)&=a^2(2-m)-(\alpha^2+\beta)-2a^2(1-m)\mathrm{nd}^2(a(x-2\alpha t)|m),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\psi_3(x,t)&=\pm a\sqrt{2m(2\alpha-1)}\,\mathrm{e}^{\mathrm{i}\eta}\,\mathrm{sn}(a(x-2\alpha t)|m),\quad \alpha>\frac{1}{2},\\ v_3(x,t)&=a^2(1-m)-(\alpha^2+\beta)-2a^2\mathrm{dn}^2(a(x-2\alpha t)|m),\end{aligned} \tag{2.24}$$

$$\begin{aligned}\psi_4(x,t)&=\pm a\sqrt{2m(2\alpha-1)}\,\mathrm{e}^{\mathrm{i}\eta}\,\mathrm{cd}(a(x-2\alpha t)|m),\quad \alpha>\frac{1}{2},\\ v_4(x,t)&=a^2(1-m)-(\alpha^2+\beta)-2a^2(1-m)\mathrm{nd}^2(a(x-2\alpha t)|m),\end{aligned} \tag{2.25}$$

and the combined JEF solutions

$$\begin{aligned}\psi_5(x,t)&=\pm a\sqrt{\frac{1-2\alpha}{2}}\left[\mathrm{dn}(a\xi|m)+\mathrm{i}\sqrt{m}\,\mathrm{sn}(a\xi|m)\right]\mathrm{e}^{\mathrm{i}\eta},\quad \alpha<\frac{1}{2}\\ v_5(x,t)&=a^2(1-m)-(\alpha^2+\beta)-a^2\left[\mathrm{dn}^2(a\xi|m)\pm\mathrm{i}\sqrt{m}\,\mathrm{sn}(a\xi|m)\mathrm{dn}(a\xi|m)\right],\end{aligned} \tag{2.26}$$

$$\begin{aligned}\psi_6&=\pm a\sqrt{\frac{1-2\alpha}{2}}[\sqrt{1-m}\,\mathrm{nd}(a\xi|m)+\mathrm{i}\sqrt{m}\,\mathrm{cd}(a\xi|m)]\,\mathrm{e}^{\mathrm{i}\eta},\quad \alpha<\frac{1}{2}\\ v_6&=a^2(1-m)-(\alpha^2+\beta)-a^2\left[(1-m)\mathrm{nd}^2(a\xi|m)\pm\mathrm{i}\sqrt{m(1-m)}\,\mathrm{cd}(a\xi|m)\mathrm{nd}(a\xi|m)\right],\end{aligned} \tag{2.27}$$

with $\xi=x-2\alpha t$.

With $m\longrightarrow 1$ in (2.22) and (2.24), the solitary wave solutions to (2.15) given in [20] are recovered. If $m\longrightarrow 1$, then (2.26) becomes

$$\begin{aligned}\psi_5(x,t)&=\pm a\sqrt{\frac{1-2\alpha}{2}}\left[\mathrm{sech}(a\xi)+\mathrm{i}\tanh(a\xi)\right]\mathrm{e}^{\mathrm{i}\eta},\quad \alpha<\frac{1}{2},\\ v_5(x,t)&=-(\alpha^2+\beta)-a^2\left[\mathrm{sech}^2(a\xi)\pm\mathrm{i}\tanh(a\xi)\,\mathrm{sech}(a\xi)\right].\end{aligned} \tag{2.28}$$

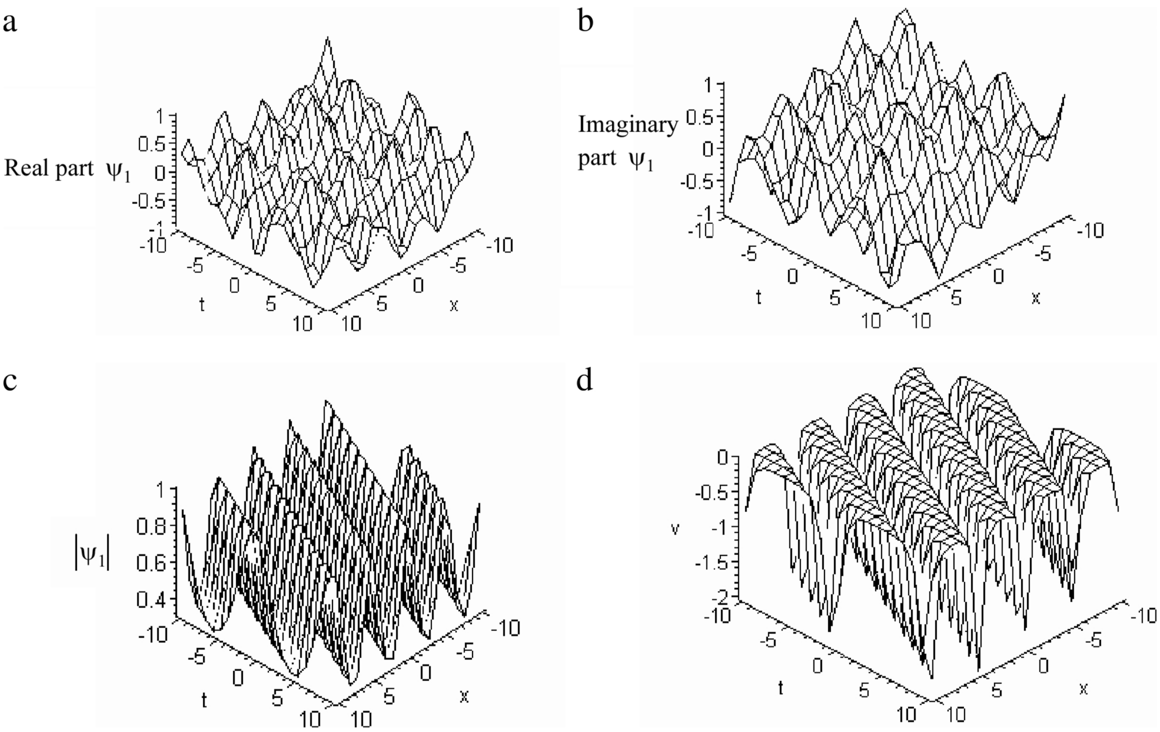

Fig. 2. The JEF solution (2.22) with parameters $\beta = a = 1$, $\alpha = 0.25$, $m = 0.95$; (a): the real part of ψ_1; (b): the imaginary part of ψ_1; (c): the modulus of ψ_1; (d): the JEF solution v_1.

The solutions (2.22) – (2.25) are the same as the results obtained in [20] by means of the F-expansion method. Compared with the solutions given in [20], here we further find some new solutions (2.26), (2.27), and (2.28).

To show the properties of the JEF solutions of (2.15), we draw plots for the solution ψ_1 and v_1 (see Fig. 2).

2.3. The DS Equations

The DS equations [21, 22] read

$$\begin{aligned} &\mathrm{i}u_t + \alpha u_{xx} + u_{yy} + \beta |u|^2 u - 2uv = 0, \\ &\alpha v_{xx} - v_{yy} - \alpha\beta(|u|^2)_{xx} = 0, \end{aligned} \tag{2.29}$$

where $\alpha = \pm 1$ and β is a constant. Equations (2.29) with $\alpha = 1$ and $\alpha = -1$ are called the DSI and DSII equations, respectively. These equations were introduced in order to discuss the instability of uniform trains of weakly nonlinear water waves in two-dimensional space. The exact periodic wave solutions of the DS equations have been studied in [15, 22, 23]. To seek travelling wave solutions, one introduces the following transformations:

$$\begin{aligned} &u(x,y,t) = \mathrm{e}^{\mathrm{i}\theta} U(\xi), \quad v(x,y,t) = V(\xi), \\ &\theta = px + qy + rt, \quad \xi = x + ly - \omega t, \end{aligned} \tag{2.30}$$

where p, q, r and l are real constants. Substituting (2.30) into (2.29), we find that $\omega = 2(\alpha p + lq)$ and U, V satisfy the following coupled system of ODEs:

$$\begin{aligned} &-(\alpha p^2 + q^2 + r)U + (\alpha + l^2)U'' + \beta U^3 - 2UV = 0, \\ &(\alpha - l^2)V'' - \alpha\beta(U^2)'' = 0. \end{aligned} \tag{2.31}$$

By means of the mixed dn-sn method and using the same procedure as above, we obtain the following exact solutions of the DS equations:

$$\begin{aligned} &u_1 = \pm a\sqrt{2(l^2 - \alpha)/\beta}\, \mathrm{e}^{\mathrm{i}\theta} \mathrm{dn}(a\xi|m), \quad \beta(l^2 - \alpha) > 0, \\ &v_1 = \frac{1}{2}[a^2(2 - m)(\alpha + l^2) - (\alpha p^2 + q^2 + r)] - 2\alpha a^2 \mathrm{dn}^2(a\xi|m), \end{aligned} \tag{2.32}$$

$$\begin{aligned} u_2 &= \pm a\sqrt{2(1-m)(l^2-\alpha)/\beta}\, \mathrm{e}^{\mathrm{i}\theta}\mathrm{nd}(a\xi|m), \quad \beta(l^2-\alpha)>0,\\ v_2 &= \frac{1}{2}\left[a^2(2-m)(\alpha+l^2)-(\alpha p^2+q^2+r)\right]-2\alpha(1-m)a^2\mathrm{nd}^2(a\xi|m), \end{aligned} \tag{2.33}$$

$$\begin{aligned} u_3 &= \pm a\sqrt{2m(\alpha-l^2)/\beta}\, \mathrm{e}^{\mathrm{i}\theta}\mathrm{sn}(a\xi|m), \quad \beta(\alpha-l^2)>0,\\ v_3 &= \frac{1}{2}\left[a^2(1-m)(\alpha+l^2)+2a^2(\alpha-l^2)-(\alpha p^2+q^2+r)\right]-2\alpha a^2\mathrm{dn}^2(a\xi|m), \end{aligned} \tag{2.34}$$

$$\begin{aligned} u_4 &= \pm a\sqrt{2m(\alpha-l^2)/\beta}\, \mathrm{e}^{\mathrm{i}\theta}\mathrm{cd}(a\xi|m), \quad \beta(\alpha-l^2)>0,\\ v_4 &= \frac{1}{2}\left[a^2(1-m)(\alpha+l^2)+2a^2(\alpha-l^2)-(\alpha p^2+q^2+r)\right]-2\alpha(1-m)a^2\mathrm{nd}^2(a\xi|m), \end{aligned} \tag{2.35}$$

$$\begin{aligned} u_5 &= \pm a\sqrt{\frac{l^2-\alpha}{2\beta}}\, \mathrm{e}^{\mathrm{i}\theta}\left[\mathrm{dn}(a\xi|m)+\mathrm{i}\sqrt{m}\mathrm{sn}(a\xi|m)\right], \quad \beta(l^2-\alpha)>0,\\ v_5 &= L-\alpha a^2\left[\mathrm{dn}^2(a\xi|m)\pm\mathrm{i}\sqrt{m}\mathrm{dn}(a\xi|m)\mathrm{sn}(a\xi|m)\right], \end{aligned} \tag{2.36}$$

$$\begin{aligned} u_6 &= \pm a\sqrt{\frac{l^2-\alpha}{2\beta}}\, \mathrm{e}^{\mathrm{i}\theta}\left[\sqrt{1-m}\mathrm{nd}(a\xi|m)+\mathrm{i}\sqrt{m}\mathrm{cd}(a\xi|m)\right], \quad \beta(l^2-\alpha)>0,\\ v_6 &= L-\alpha a^2\left[(1-m)\mathrm{nd}^2(a\xi|m)\pm\mathrm{i}\sqrt{m(1-m)}\mathrm{nd}(a\xi|m)\mathrm{cd}(a\xi|m)\right], \end{aligned} \tag{2.37}$$

with $L=\frac{1}{2}[a^2(1-m)(\alpha+l^2)+\frac{a^2}{2}(\alpha-l^2)-(\alpha p^2+q^2+r)]$, $\xi=x+ly-2(\alpha p+lq)t$ and a, p, q, r, and l being arbitrary constants. If $m\longrightarrow 1$, then (2.32) and (2.34) become the solitary wave solutions of the DS equations

$$\begin{aligned} u_1(x,y,t) &= \pm a\sqrt{2(l^2-\alpha)/\beta}\, \mathrm{e}^{\mathrm{i}\theta}\mathrm{sech}(a\xi), \quad \beta(l^2-\alpha)>0,\\ v_1(x,y,t) &= \frac{1}{2}\left[a^2(\alpha+l^2)-(\alpha p^2+q^2+r)\right]-2\alpha a^2\mathrm{sech}^2(a\xi),\\ u_3(x,y,t) &= \pm a\sqrt{2(\alpha-l^2)/\beta}\, \mathrm{e}^{\mathrm{i}\theta}\tanh(a\xi), \quad \beta(\alpha-l^2)>0,\\ v_3(x,y,t) &= \frac{1}{2}\left[2a^2(\alpha-l^2)-(\alpha p^2+q^2+r)\right]-2\alpha a^2\mathrm{sech}^2(a\xi), \end{aligned} \tag{2.38}$$

and the solutions (2.36) degenerate to new exact solutions

$$\begin{aligned} u_5(x,y,t) &= \pm a\sqrt{\frac{l^2-\alpha}{2\beta}}\, \mathrm{e}^{\mathrm{i}\theta}[\mathrm{sech}(a\xi)+\mathrm{i}\tanh(a\xi)], \quad \beta(l^2-\alpha)>0,\\ v_5(x,y,t) &= \frac{1}{2}\left[\frac{a^2}{2}(\alpha-l^2)-(\alpha p^2+q^2+r)\right]-\alpha a^2\left[\mathrm{sech}^2 a\xi\pm\mathrm{i}\,\mathrm{sech}a\xi\tanh a\xi\right]. \end{aligned} \tag{2.39}$$

The solutions (2.32) and (2.34) coincide with the solutions given in [22] by means of the mapping method. Compared with the results in [22], here we further find some new solutions (2.33), (2.35) – (2.37) and (2.39). The plots for the solutions of DS equations are not given here since they are similar to those in Figure 2.

3. Conclusion

We have extended the mixed dn-sn method to seek exact solutions of coupled nonlinear evolution equations of mathematical physics. The JEF solutions to the coupled mKdV equations, the long-short-wave interaction system and DS equations are obtained by using this method. We believe, to the best of our knowledge, that the combined JEF solutions to these coupled equations are new. When $m \longrightarrow 1$, the solitary wave solutions are also found. On using our method, we recovered not only the known solutions but also found new exact solutions of such coupled equations. The obtained solutions include periodic wave solutions, combined JEF solutions and solitary wave solutions. The properties of some JEF solutions are shown in Figures 1 and 2.

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